Figure 13 shows the graph of the food-price index function I of Example 7. This model is based on the data points shown.

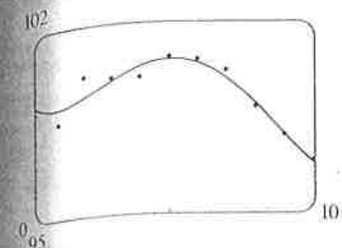


FIGURE 13

EXAMPLE 7 ■ A model for the food-price index (the price of a representative "basket" of foods) between 1984 and 1994 is given by the function

$$I(t) = 0.00009045t^5 + 0.001438t^4 - 0.06561t^3 + 0.4598t^2 - 0.6270t + 99.33$$

where t is measured in years since midyear 1984, so $0 \leqslant t \leqslant 10$, and $I(t)$ is measured in 1987 dollars and scaled such that $I(3) = 100$. Estimate the times when food was cheapest and most expensive during the period 1984–1994.

SOLUTION We apply the Closed Interval Method to the continuous function I on $[0, 10]$. Its derivative is

$$I'(t) = 0.00045225t^4 + 0.005752t^3 - 0.19683t^2 + 0.9196t - 0.6270$$

Since I' exists for all t, the only critical numbers of I occur when $I'(t) = 0$. We use a root-finder on a computer algebra system (or a graphing device) to find that $I'(t) = 0$ when $t \approx -29.7186, 0.8231, 5.1309$, or 11.0459, but only the second and third roots lie in the interval $[0, 10]$. The values of I at these critical numbers are

$$I(0.8231) \approx 99.09 \qquad \text{and} \qquad I(5.1309) \approx 100.67$$

The values of I at the endpoints of the interval are

$$I(0) = 99.33 \qquad I(10) \approx 96.86$$

Comparing these four numbers, we see that food was most expensive at $t \approx 5.1309$ (corresponding roughly to August, 1989) and cheapest at $t = 10$ (midyear 1994). ■

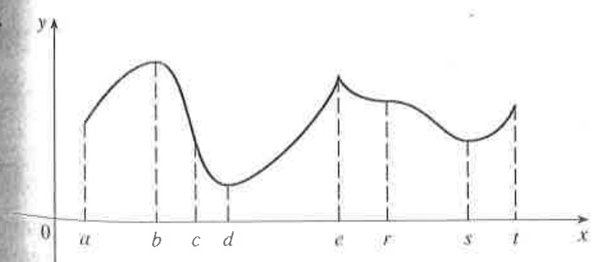
4.2 Exercises

1. Explain the difference between an absolute minimum and a local minimum.

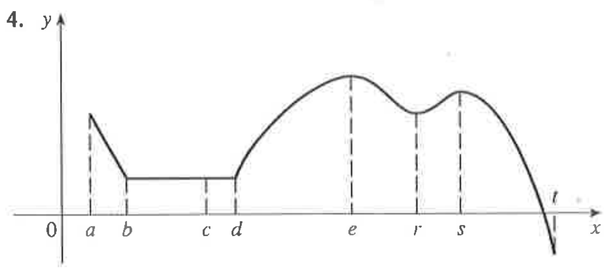
2. Suppose f is a continuous function defined on a closed interval $[a, b]$.
   (a) What theorem guarantees the existence of an absolute maximum value and an absolute minimum value for f?
   (b) What steps would you take to find those maximum and minimum values?

3–4 ■ For each of the numbers a, b, c, d, e, r, s, and t, state whether the function whose graph is shown has an absolute maximum or minimum, a local maximum or minimum, or neither a maximum nor a minimum.

3.

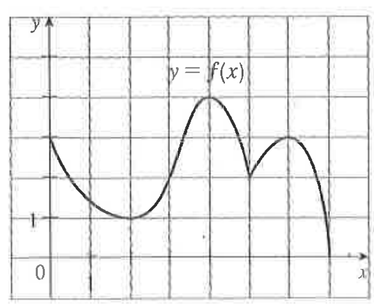


4.

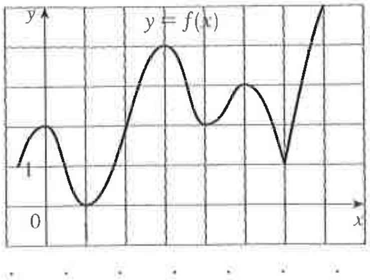


5–6 ■ Use the graph to state the absolute and local maximum and minimum values of the function.

5.



6.



7–10 ■ Sketch the graph of a function f that is continuous on $[0, 3]$ and has the given properties.

7. Absolute maximum at 0, absolute minimum at 3, local minimum at 1, local maximum at 2

8. Absolute maximum at 1, absolute minimum at 2

9. 2 is a critical number, but f has no local maximum or minimum

10. Absolute minimum at 0, absolute maximum at 2, local maxima at 1 and 2, local minimum at 1.5

11. (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
(b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
(c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.

12. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no local maximum.
(b) Sketch the graph of a function on $[-1, 2]$ that has a local maximum but no absolute maximum.

13. (a) Sketch the graph of a function on $[-1, 2]$ that has an absolute maximum but no absolute minimum.
(b) Sketch the graph of a function on $[-1, 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.

14. (a) Sketch the graph of a function that has two local maxima, one local minimum, and no absolute minimum.
(b) Sketch the graph of a function that has three local minima, two local maxima, and seven critical numbers.

15–24 ■ Find the absolute and local maximum and minimum values of f. Begin by sketching its graph by hand. (Use the graphs and transformations of Section 1.2.)

15. $f(x) = 1 + 2x, \quad x \geq -1$

16. $f(x) = 1 - x^2, \quad 0 < x \leq 1$

17. $f(x) = 1 - x^2, \quad -2 \leq x \leq 1$

18. $f(t) = 1/t, \quad 0 < t < 1$

19. $f(\theta) = \sin\theta, \quad -2\pi \leq \theta \leq 2\pi$

20. $f(\theta) = \tan\theta, \quad -\pi/4 \leq \theta < \pi/2$

21. $f(x) = x^5$

22. $f(x) = 2 - x^4$

23. $f(x) = 1 - e^{-x}, \quad x \geq 0$

24. $f(x) = \begin{cases} x^2 & \text{if } -1 \leq x < 0 \\ 2 - x^2 & \text{if } 0 \leq x \leq 1 \end{cases}$

25–36 ■ Find the critical numbers of the function.

25. $f(x) = 4x^3 - 9x^2 - 12x + 3$

26. $f(t) = t^3 + 6t^2 + 3t - 1$

27. $s(t) = t^4 + 4t^3 + 2t^2$

28. $g(x) = |x + 1|$

29. $f(r) = \dfrac{r}{r^2 + 1}$

30. $f(z) = \dfrac{z + 1}{z^2 + z + 1}$

31. $F(x) = x^{4/5}(x - 4)^2$

32. $G(x) = \sqrt[3]{x^2 - x}$

33. $f(\theta) = \sin^2(2\theta)$

34. $g(\theta) = \theta + \sin\theta$

35. $f(x) = x\ln x$

36. $f(x) = xe^{2x}$

37–46 ■ Find the absolute maximum and absolute minimum values of f on the given interval.

37. $f(x) = x^2 - 2x + 2, \quad [0, 3]$

38 $f(x) = x^3 - 12x + 1, \quad [-3, 5]$

39. $f(x) = 3x^5 - 5x^3 - 1, \quad [-2, 2]$

40. $f(x) = \sqrt{9 - x^2}, \quad [-1, 2]$

41. $f(x) = x^2 + 2/x, \quad [\frac{1}{2}, 2]$

42. $f(x) = \dfrac{x}{x + 1}, \quad [1, 2]$

43. $f(x) = \sin x + \cos x, \quad [0, \pi/3]$

44. $f(x) = x - 2\cos x, \quad [-\pi, \pi]$

45. $f(x) = xe^{-x}, \quad [0, 2]$

46. $f(x) = (\ln x)/x, \quad [1, 3]$

47–50 ■
(a) Use a graph to estimate the absolute maximum and minimum values of the function to two decimal places.
(b) Use calculus to find the exact maximum and minimum values.

47. $f(x) = x^3 - 8x + 1, \quad -3 \leq x \leq 3$

48. $f(x) = e^{x^3 - x}, \quad -1 \leq x \leq 0$

49. $f(x) = x\sqrt{x - x^2}$

50. $f(x) = (\cos x)/(2 + \sin x), \quad 0 \leq x \leq 2\pi$

51. Between 0 °C and 30 °C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given